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The thermodynamic limit and the replica method for short-range random systems

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Abstract. Short-range spin systems with random interactions are considered. A simple proof is given showing that the free energy of almost every sample converges to the average free energy in the thermodynamic limit. A stronger criterion, *thermodynamic convergence*, is also demonstrated. This implies that the $N \to \infty$ and $n \to 0$ limits may be interchanged in the replica method.

1. Introduction

There is considerable current interest in random systems, which are usually described by a Hamiltonian H that depends on some quenched random variables as well as dynamical quantities. For a particular configuration $\{J\}$ of the random variables the corresponding free energy per site $f(\beta, \{J\})$ in the thermodynamic limit is given by

$$-\beta f(\beta, \{J\}) = \lim_{N\to\infty} N^{-1} \ln Z_N(\beta, \{J\}) \tag{1}$$

where

$$Z_N(\beta, \{J\}) = \mathrm{Tr} \exp(-\beta H).$$

Here $Z_N(\beta, \{J\})$, usually abbreviated to Z_N, is the partition function for the configuration $\{J\}$ at inverse temperature $\beta = 1/kT$, and N is the number of sites or particles. Generally one is only able to compute the *average* free energy in the thermodynamic limit,

$$-\beta f(\beta) = \lim_{N\to\infty} \langle N^{-1} \ln Z_N \rangle. \tag{2}$$

We always use $\langle \ldots \rangle$ to denote averaging with respect to the configuration $\{J\}$.

The average $f(\beta)$ may be computed using the replica method. Defining for real n

$$\phi_N(n) = N^{-1} \ln\langle Z_N^n \rangle \tag{3}$$

we see that

$$\phi_{N}'(0) = \langle N^{-1} \ln Z_N \rangle$$

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(with the prime denoting differentiation with respect to n), and thus

$$-\beta f(\beta)=\lim_{N\to\infty}\phi_N'(0). \tag{4}$$

In practice one needs the thermodynamic limit to compute the partition function, so one actually computes $\phi'(0)$ where

$$\phi(n)=\lim_{N\to\infty}\phi_N(n). \tag{5}$$

$\phi'(0)$ is equal to $-\beta f(\beta)$ *if* the limits on N and n (differentiation) are interchangeable.

In a previous paper (van Hemmen and Palmer 1979) we studied questions of the existence and interchange of limits, with emphasis on the Sherrington-Kirkpatrick (1975) long-range spin glass model. Naive application of the replica method, computing $\phi'(0)$ from an 'obvious' extension from rigorously known values $\phi(n)$ at positive integer n, gives an incorrect result for $-\beta f(\beta)$. We were able to show that the obvious extension is definitely wrong; if it had been correct the N and n limits would have been interchangeable. Recent theories (e.g. Parisi 1979) construct different extensions by 'replica symmetry breaking'. Note however that there is *no* replica symmetry breaking for positive integer n.

In this paper we consider short-range lattice models. We first study the thermodynamic limit and show that the limits (1) and (4) exist and *are equal with probability one*. Thus, almost every configuration $\{J\}$ gives the same result in the thermodynamic limit. Introducing a novel criterion, *thermodynamic convergence*, we then show that the N and n limits may be interchanged, so that $-\beta f(\beta)=\phi'(0)$ for these models.

The key to our proofs is that systems with short-range interactions may be decomposed into many 'weakly' interacting subsystems which each already approximate the macroscopic ($N\to\infty$) system fairly well. Brout (1959) seems to have been the first to realise that this property implies that the typical free energy is close to the average $f(\beta)$.

We illustrate the problem with a very simple example in § 2 before turning to the thermodynamic limit in §§ 3 and 4, and the replica method in § 5.

2. A simple example

We consider a one-dimensional Ising chain with random nearest-neighbour interactions J_i, described by

$$H=-\sum_{i=1}^{N} J_iS_iS_{i+1}.$$

The J_i are independent identically distributed (IID) random variables with mean $\langle J\rangle$ and variance $\langle J^2\rangle-\langle J\rangle^2$. Each S_i is ±1, and we assume free boundary conditions. There are N interactions and $N+1$ spins; we call the system size N in (1)–(5) for convenience, the difference disappearing in the thermodynamic limit.

We may write

$$\exp(\beta H)=\prod_{i=1}^{N}\cosh\beta J_i\prod_{i=1}^{N}(1+S_iS_{i+1}\tanh\beta J_i)$$

and because every term (except 1) in the expansion of the second product contains some S_i to an odd power, we obtain simply

$$Z_N = \prod_{i=1}^{N} (2 \cosh \beta J_i). \tag{6}$$

Thus

$$-\beta f(\beta, \{J\}) = \lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} \ln(2 \cosh \beta J_i)$$

which involves the mean of N IID random variables $\ln(2 \cosh \beta J_i)$ with $N \to \infty$. By the strong law of large numbers (Feller 1968), this is equal in the limit $N \to \infty$ to the common population mean $\langle \ln(2 \cosh \beta J)\rangle$ *almost everywhere*. 'Almost everywhere' (AE) means that the set of configurations $\{J\}$ for which the result does *not* hold has probability (i.e. measure) zero. We therefore have

$$-\beta f(\beta, \{J\}) \overset{\text{AE}}{=} \langle \ln(2 \cosh \beta J)\rangle$$

so that the free energy per site is independent of the configuration $\{J\}$ almost everywhere in the thermodynamic limit. This feature is characteristic of systems which can be broken into many (nearly) independent parts, as we shall see for short-range systems in the next section.

Turning to the replica method, we can evaluate (3) explicitly for all real n by using (6) and the fact that the J_i are IID:

$$\phi_N(n) = N^{-1} \ln\left\langle \prod_{i=1}^{N} (2 \cosh \beta J_i)^n \right\rangle = \ln\langle (2 \cosh \beta J)^n \rangle.$$

The thermodynamic limit in (4) and (5) is then trivial and we conclude

$$-\beta f(\beta) = \phi'(0) = \langle \ln(2 \cosh \beta J)\rangle \overset{\text{AE}}{=} -\beta f(\beta, \{J\}).$$

Thus in this special case the N and n limits *may* be interchanged. One is inclined to think that the almost everywhere convergence is sufficient to guarantee this interchange in general. In §§ 4 and 5 we show that a slightly stronger 'thermodynamic convergence' criterion pertains in short-range models and does allow the interchange.

3. Almost everywhere convergence

We consider a quantum spin system on a d-dimensional cubic lattice of N sites, described by the Hamiltonian†

$$H = -\sum_{ij} J_{ij} S_i S_j - \sum_i h_i S_i.$$

The J_{ij}, $1 \leqslant i < j \leqslant N$, and the h_i, $1 \leqslant i \leqslant N$, are independent random variables. The distribution of the J_{ij} only depends on $(i-j)$, whereas the h_i are identically distributed. To simplify the argument we take $|J_{ij}| \leqslant B$, and $h_i = 0$; the former assumption is technical, the latter convenient. Furthermore we assume $J_{ij} = 0$ if $|i-j| > \frac{1}{2}r$.

† For the sake of definiteness one may imagine that this Hamiltonian describes an Ising spin system, but the reasoning below applies equally well to Heisenberg and more general quantum models.

In this section we prove (*a*) that the limit in (2) exists, so $\langle N^{-1} \ln Z_N \rangle \to -\beta f(\beta)$ as $N \to \infty$, and (*b*) that the limit in (1) exists, and is equal to $-\beta f(\beta)$, almost everywhere, so $N^{-1} \ln Z_N \xrightarrow{\text{AE}} 1 - \beta f(\beta)$ as $N \to \infty$. The problem has been considered before by Vuillermot (1977), but the present arguments are simpler and will be used again in the next section.

We take a sequence of cubes of increasing size N which finally fill the whole space. The side of a cube is taken to be mk, so that $N = (mk)^d$. We consider a division into $K = k^d$ subcubes of side m and $M = m^d$ sites each. The subcubes interact with each other only in the 'corridors' at their boundaries; cf Huang (1963, appendix C). As the range of the interaction is $\frac{1}{2}r$, the width of a corridor is r. Taking advantage of the Bogoliubov–Peierls inequality (Ruelle 1969, equation (2.15))

$$|\ln \operatorname{Tr} \exp(A) - \ln \operatorname{Tr} \exp(B)| \leq \|A - B\| \tag{7}$$

we may write

$$\ln Z_N = \sum_{i=1}^{K} \ln Z_M(i) + R_N$$

where R_N takes into account the interactions between different subcubes. Because the number of points in the corridors is less than $2dKm^{d-1}r$, we have

$$R_N \leq ABKm^{d-1}r^{d+1} \tag{8}$$

with A a geometrical constant. Hence

$$\left| N^{-1} \ln Z_N - K^{-1} \sum_{i=1}^{K} M^{-1} \ln Z_M(i) \right| \leq R_N/N = C/m \tag{9}$$

where the constant C depends only on the geometry and the interaction range and strength.

The second term of (9) is the mean of K IID random variables $M^{-1} \ln Z_M(i)$ and thus, by the strong law of large numbers (Feller 1968), converges almost everywhere as $K \to \infty$ to the common population mean $\langle M^{-1} \ln Z_M \rangle$. Thus, taking the limit $K \to \infty$, and hence $N \to \infty$, we find for fixed M and almost every configuration $\{J\}$

$$-C/m + \langle M^{-1} \ln Z_M \rangle \leq \liminf_{N\to\infty} N^{-1} \ln Z_N \leq \limsup_{N\to\infty} N^{-1} \ln Z_N \leq \langle M^{-1} \ln Z_M \rangle + C/m. \tag{10}$$

This result does not depend on the specific sequence of cubes used; a sequence with sides $mk + a$, $N = (mk + a)^d$, $1 \leq a < m$ gives the very same estimate because the extra contribution to R_N/N from the outer surface goes to zero as $K \to \infty$.

Whatever N, $N^{-1} \ln Z_N$ is uniformly bounded; this follows directly from (7) with $A := H$, $B := 0$. The same therefore holds true for $\langle N^{-1} \ln Z_N \rangle$, so Fatou's lemma (Halmos 1950, p 113) implies

$$\limsup_{N\to\infty} \langle N^{-1} \ln Z_N \rangle \leq \left\langle \limsup_{N\to\infty} N^{-1} \ln Z_N \right\rangle \leq \langle M^{-1} \ln Z_M \rangle + C/m$$

with the second inequality coming from (10). By sending M and m to infinity we arrive at

$$\limsup_{N\to\infty} \langle N^{-1} \ln Z_N \rangle \leq \liminf_{N\to\infty} \langle N^{-1} \ln Z_N \rangle$$

so the limit in (2)

$$-\beta f(\beta) = \lim_{N\to\infty} \langle N^{-1} \ln Z_N \rangle \tag{11}$$

exists as claimed.

Returning to (10) we again send m and M to infinity to conclude that, for almost every configuration $\{J\}$,

$$-\beta f(\beta, \{J\}) = \lim_{N\to\infty} N^{-1} \ln Z_N \tag{12}$$

exists and is equal to $-\beta f(\beta)$ from (11). This justifies the usual procedure of computing (11) in place of (12); one may average without altering the result. The mean, typical, and most probable values of thermodynamic quantities are thus identical.

Two extensions deserve mention. Firstly, the thermodynamic limit for a sequence of fairly arbitrarily shaped volumes (à la van Hove, say) may be reduced to the present case by a standard argument (Ruelle 1969). Secondly, the result remains valid for *un*bounded J_{ij} if the first moment of $|J_{ij}|$ is finite, as is true of all physically relevant distributions except the Lorentzian. In equation (8) B must be replaced by $\Sigma' |J_{ij}|/\Sigma' 1$, where Σ' refers to pairs of interacting sites within the corridors, and our proof requires B to remain bounded as first K and then M are sent to infinity. The first is guaranteed by the finite moment, but the second involves a measure-theoretic subtlety because the corridors change with M. It goes through, however, because all the operations are countable.

The conclusion that averaging does not alter the result is applicable to thermodynamic quantities derivable from the free energy. It may not be true, however, for local quantities such as two-point correlation functions, whose value typically *does* depend on the specific configuration $\{J\}$. For example, in many spin glass models one takes an even distribution for J_{ij}, which allows a local gauge transformation

$$S_i \to -S_i \qquad J_{ij} \to -J_{ij}$$

(for fixed i, and all j connected to i) within an average. Thus all *averaged* m-point correlation functions are zero (cf Chalupa 1977). In fact one can show an even stronger result. Take a *specific* sample and consider the correlation function $\langle S(r)S(r+R)\rangle_\beta$ with R fixed. Here $\langle \ldots \rangle_\beta$ is a thermal average, which we assume to exist ($N \to \infty$). Whatever the correlation function its *spatial* average vanishes with probability one. To see this, note that for instance

$$\lim_{N\to\infty} N^{-1} \sum_r \langle S(r)S(r+R)\rangle_\beta \stackrel{\mathrm{AE}}{=} \langle\langle S(0)S(R)\rangle_\beta\rangle$$

by the ergodic theorem (cf van Hemmen 1977). The right-hand side is zero for a symmetric J_{ij} distribution, as just discussed. Despite these results we do not expect all spin–spin correlations to vanish in a particular sample, so we must conclude that averaging over randomness, or even spatially, in general does not provide pertinent information.

4. Thermodynamic convergence

To prove the interchange of limits in the replica method we need a stronger notion of convergence than the almost everywhere result of the last section.

Let P be a probability measure, and $P\{A\}$ be the probability that an event A occurs. Technically, P is defined on a probability space Ω whose elements are denoted by ω, and $A = \{\omega \in A\}$. We say that $N^{-1}W_N$ converges to α *thermodynamically* (Ellis 1981), and write $N^{-1}W_N \xrightarrow{\text{th}} \alpha$, if for any $\delta > 0$ we can find a constant $c = c(\delta) > 0$ such that for all sufficiently large N

$$P\{|N^{-1}W_N - \alpha| \geq \delta\} \leq \exp(-cN). \tag{13}$$

We now prove that thermodynamic convergence implies almost everywhere convergence; the reverse is not true. Let $1_{N,\delta}$ be the indicator function of the event $\{|N^{-1}W_N - \alpha| \geq \delta\}$; it is either zero or one, and in any case non-negative. Since by (13)

$$\int \mathrm{d}P \sum_{N=1}^{\infty} 1_{N,\delta} = \sum_{N=1}^{\infty} \int \mathrm{d}P \, 1_{N,\delta} \leq \sum_{N=1}^{\infty} \exp(-cN) < \infty$$

we find, for almost every configuration ω

$$\sum_{N=1}^{\infty} 1_{N,\delta}(\omega) < \infty$$

which implies the existence of a last non-zero term. Thus for almost every ω there exists an $N(\omega)$ such that $|N^{-1}W_N(\omega) - \alpha| < \delta$ for $N > N(\omega)$. We can repeat this whole argument for a sequence of δ's tending to zero, and hence conclude that $N^{-1}W_N$ converges to α almost everywhere.

To demonstrate thermodynamic convergence in our short-range models we first need an important result corresponding to the strong law of large numbers used previously for the almost everywhere convergence.

Lemma. If X_i, $1 \leq i \leq K$ are K independent identically distributed random variables with mean $\langle X \rangle$ and differentiable moment generating function $\langle \exp(tX) \rangle$, then

$$K^{-1} \sum_{i=1}^{K} X_i \xrightarrow{\text{th}} \langle X \rangle$$

as $K \to \infty$.

Proof. Let $S_K = \sum_{i=1}^{K} (X_i - \langle X \rangle)$. Then, for any $t > 0$,

$$\begin{aligned} P\{K^{-1}S_K \geq \delta\} = P\{S_K \geq K\delta\} &= \int_{\{S_K \geq K\delta\}} \mathrm{d}P \leq \int_{\{S_K \geq K\delta\}} \mathrm{d}P \exp[t(S_K - K\delta)] \\ &\leq \exp(-tK\delta) \int_{\Omega} \mathrm{d}P \exp(tS_K) \\ &= \exp(-tK\delta)\Psi(t)^K = \exp(-tK[\delta - t^{-1} \ln \Psi(t)]) \end{aligned}$$

where $\Psi(t) = \langle \exp[t(X - \langle X \rangle)] \rangle$. Now $\Psi(t)$ is differentiable, and $\Psi(0) = 1$, $\Psi'(0) = 0$. Hence, by taking t sufficiently small, $\delta - t^{-1} \ln \Psi(t)$ can be made positive. The case $K^{-1}S_K \leq -\delta$ can be handled similarly, so the result is established.

Returning to the quantum lattice system of the last section, let us split up the error term by the triangle inequality:

$$|N^{-1}\ln Z_N - \lim_{N\to\infty}\langle N^{-1}\ln Z_N\rangle| \leq \left|N^{-1}\ln Z_N - K^{-1}\sum_{i=1}^{K} M^{-1}2\ln Z_M(i)\right|$$

$$+\left|K^{-1}\sum_{i=1}^{K} M^{-1}\ln Z_M(i) - \langle M^{-1}\ln Z_M\rangle\right|$$

$$+|\langle M^{-1}\ln Z_M\rangle - \lim_{N\to\infty}\langle N^{-1}\ln Z_N\rangle|.$$

Whatever $\{J\}$, for large enough M the first and last terms on the right-hand side can each be made no larger than $\frac{1}{3}\delta$ for any $\delta > 0$; the first by (9), and the last because the limit in (11) exists. Our lemma then gives

$$P\{|N^{-1}\ln Z_N - \lim_{N\to\infty}\langle N^{-1}\ln Z_N\rangle| \geq \delta\}$$

$$\leq P\left\{\left|K^{-1}\sum_{i=1}^{K} M^{-1}\ln Z_M(i) - \langle M^{-1}\ln Z_M\rangle\right| \geq \tfrac{1}{3}\delta\right\}$$

$$\leq \exp(-cK)$$

so that

$$N^{-1}\ln Z_N \xrightarrow{\text{th}} \lim_{N\to\infty}\langle N^{-1}\ln Z_N\rangle = -\beta f(\beta)$$

as K, and thus N, tends to infinity.

5. The replica method

Given the thermodynamic convergence of $N^{-1}\ln Z_N$, the validity of the interchange of the N and n limits in the replica method is easily obtained from the following theorem (Ellis 1981).

Theorem. The following two statements are equivalent

(i) There exists a real α such that $N^{-1}W_N \xrightarrow{\text{th}} \alpha$.
(ii) $\phi(n)$ is differentiable at $n = 0$ and $\phi'(0) = \alpha$, where

$$\phi(n) = \lim_{N\to\infty} N^{-1}\ln\langle\exp(nW_N)\rangle.$$

This theorem is proved in the appendix.

In the present case $W_N := \ln Z_N$, $\phi(n)$ is the replica $\phi(n)$ of (3) and (5), and we have established (i) in the preceding section, with $\alpha = -\beta f(\beta)$. Thus indeed $\phi'(0) = -\beta f(\beta)$ and the limits are interchangeable.

Besides the interchange of limits, application of the replica method involves actually computing $\phi(n)$ in the neighbourhood of $n = 0$. The usual approach is the computation of $\phi(n)$ at positive integer n followed by a suitable extension to $n = 0$. As yet there is no known criterion which enables one to select a *unique* extension. The Sherrington–

Kirkpatrick (1975) *long*-range model provides an important example where an 'obvious' extension is certainly wrong (van Hemmen and Palmer 1979). The situation might be more favourable for short-range models, but the problem is still unsolved.

6. Conclusion

For systems with short-range interactions we have shown that the free energy of almost every sample is equal to the averaged free energy in the thermodynamic limit, and that one may interchange the $n \to 0$ and $N \to \infty$ limits in the replica method. In general, local quantities such as correlation functions may *not* be averaged, and because the replica method presupposes averaging it has no predictive value for such quantities. Finally, it is important to emphasise that our analysis is within equilibrium statistical mechanics, assuming the Gibbs prescription. The essence of the behaviour of certain random systems, such as spin glasses, may however lie outside this prescription, in long-lived metastable states.

Appendix

We now prove the main theorem of § 5.

Theorem. The following two statements are equivalent

(i) There exists a real α such that $N^{-1} W_N \xrightarrow{\text{th}} \alpha$.

(ii) $\phi(t)$ is differentiable at $t = 0$ and $\phi'(0) = \alpha$, where

$$\phi(t) = \lim_{N\to\infty} N^{-1} \ln\langle \exp(tW_N)\rangle \equiv \lim_{N\to\infty} \phi_N(t).$$

Apart from several simplifications the proof goes back to Ellis (1981). It is not a serious restriction to assume that $\phi(t)$ exists and is finite for all real t.

(i) $\to$ (ii). Since ϕ is a limit of the convex functions ϕ_N, it is convex itself. Being convex it is continuous, and its right and left derivatives at $t = 0$, $\phi'_+(0) = \lim_{t\downarrow 0} t^{-1}\phi(t)$ and $\phi'_-(0) = \lim_{t\uparrow 0} t^{-1}\phi(t)$, exist and satisfy the inequality $\phi'_-(0) \leqslant \phi'_+(0)$. If equality holds ϕ is differentiable at zero. We shall show $\phi'_+(0) \leqslant \alpha$; the inequality $\alpha \leqslant \phi'_-(0)$ can be proved analogously. Then $\alpha \leqslant \phi'_-(0) \leqslant \phi'_+(0) \leqslant \alpha$ gives the desired result.

Writing $Y_N = N^{-1} W_N - \alpha$, we have for $0 < t < \frac{1}{2}$ (say) and $\delta > 0$

$$\begin{aligned} t^{-1}\phi_N(t) - \alpha &= (Nt)^{-1} \ln\langle \exp(tNY_N)\rangle \\ &= (Nt)^{-1} \ln\langle \exp(tNY_N)[1_{\{|Y_N|<\delta\}} + 1_{\{|Y_N|\geqslant\delta\}}]\rangle \\ &\leqslant (Nt)^{-1} \ln[\exp(t\delta N) + \exp(-\alpha t N)\langle \exp(tW_N) 1_{\{|Y_N|\geqslant\delta\}}\rangle]. \end{aligned}$$

By the Cauchy–Schwarz inequality and (i), i.e. equation (13) of the main text, we obtain for N sufficiently large,

$$\begin{aligned} \langle \exp(tW_N) 1_{\{|Y_N|\geqslant\delta\}}\rangle &\leqslant \langle \exp(2tW_N)\rangle^{1/2} P\{|N^{-1} W_N - \alpha| \geqslant \delta\}^{1/2} \\ &\leqslant \exp[\tfrac{1}{2}N\phi_N(2t) - \tfrac{1}{2}cN] \end{aligned}$$

so that

$$t^{-1}\phi_N(t)-\alpha \leqslant (Nt)^{-1}\ln\{\exp(t\delta N)+\exp[N(\tfrac{1}{2}\phi_N(2t)-\tfrac{1}{2}c-\alpha t)]\}. \tag{A1}$$

Since ϕ_N converges to ϕ *uniformly* on [0, 1] (Roberts and Varberg 1973, theorem 13E), we can choose N so large that for all $t(0\leqslant t\leqslant\frac{1}{2})$,

$$\tfrac{1}{2}\phi_N(2t)-\tfrac{1}{2}c-\alpha t\leqslant\tfrac{1}{2}\phi(2t)-\tfrac{1}{4}c-\alpha t. \tag{A2}$$

At the end t will be sent to zero. Now $c>0$, so we can make the right-hand side of (A2) less than or equal to δt by taking $t\leqslant t_0$ for a suitable t_0. Thus we find, combining (A1) and (A2),

$$t^{-1}\phi_N(t)-\alpha\leqslant\delta+(Nt)^{-1}\ln 2$$

and as $N\to\infty$

$$t^{-1}\phi(t)-\alpha\leqslant\delta.$$

Sending first t, then δ, to zero (δ was arbitrary) completes the proof.

(ii)$\to$(i). Now we are given that $\phi(t)$ is differentiable at $t=0$, with $\phi'(0)=\alpha$, and we have to estimate $P\{|N^{-1}W_N-\alpha|\geqslant\delta\}$. As in the proof of the lemma of § 4, we obtain for $t>0$,

$$P\{N^{-1}W_N\geqslant\phi'(0)+\delta\}\leqslant\exp[tN(-\delta+t^{-1}\phi_N(t)-\phi'(0))]. \tag{A3}$$

Choose a t_0 so small that $|t_0^{-1}\phi(t_0)-\phi'(0)|<\frac{1}{4}\delta$. Next choose N so large that $t_0^{-1}|\phi_N(t_0)-\phi(t_0)|\leqslant\frac{1}{4}\delta$. Then the right-hand side of (A3) can be majorised by $\exp[-(\frac{1}{2}t_0\delta)N]$. The probability $P\{N^{-1}W_N\leqslant\phi'(0)-\delta\}$ can be handled similarly, so the result is established.

The theorem above assumes $\phi(t)=\lim\phi_N(t)$ as $N\to\infty$, whatever t. We do not need so much, however, once we are concerned with physical applications. We first notice that a finite interval I containing the origin suffices for our present purposes. Next we observe that the sequence of convex functions ϕ_N has to be bounded above and below independently of N (van Hemmen and Palmer 1979, appendix B). Thus we can find a *sub*sequence that converges uniformly on I to a function ϕ (Roberts and Varberg 1973, p 20). In § 4 we have shown that (i) holds, with

$$\alpha=\lim_{N\to\infty}\langle N^{-1}\ln Z_N\rangle.$$

Hence $\phi'(0)=\alpha$. Moreover, *whatever* subsequence we take, we always get the *same* derivative at the origin; the answer does not depend on the subsequence used.

Note added in proof. We would like to correct some misprints which remained in a previous paper (van Hemmen and Palmer 1979) for reasons beyond our control.

Equation (29) should read

$$q=\frac{\displaystyle\int_{-\infty}^{\infty}\frac{dz}{(2\pi)^{1/2}}\exp(-\tfrac{1}{2}z^2)\tanh^2[(\gamma q)^{1/2}z]\cosh^n[(\gamma q)^{1/2}z]}{\displaystyle\int_{-\infty}^{\infty}\frac{dz}{(2\pi)^{1/2}}\exp(-\tfrac{1}{2}z^2)\cosh^n[(\gamma q)^{1/2}z]}. \tag{29}$$

In equation (36) $O(z^4)$ has to be replaced by $o(z^4)$, and λ_{ij} in (38) is given by $\Sigma_{\alpha=1}^{n}\boldsymbol{S}_\alpha(i)\boldsymbol{S}_\alpha(j)$. The vector $\boldsymbol{q}$ in appendix A (p 576) is defined by $\boldsymbol{q}=\{q_{\alpha\beta};1\leqslant\alpha<\beta\leqslant n\}$, and one should put a comma after $\boldsymbol{q}''=(q_{12},B,B,C)$ in the lemma on p 557.

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